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The reduction of symmetrized powers of corepresentations of magnetic groups

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Abstract. The problem of reducing the symmetrized powers of the corepresentations of a magnetic group is solved by relating it to the analogous problem for representations of the unitary subgroup. Use is made of a formula given by Littlewood for the symmetrization of the direct sum of two matrices. A procedure is also given for the complete reduction of a reducible corepresentation which assumes that the unitary part has already been completely reduced to canonical form.

1. Introduction

Magnetic groups, which are groups containing both unitary and antiunitary elements, have found an important place in the study of magnetic materials. Since the quantum-mechanical time-reversal operator is antiunitary these groups also arise in other branches of physics. The literature on magnetic groups and their corepresentations, the latter being a construction introduced by Wigner, is now rather extensive, but fairly complete studies and bibliographies may be found in the following books and articles: Wigner (1959), Bradley and Davies (1968), Bradley and Cracknell (1972), Opechowski and Guccione (1965).

Although corepresentation theory has been well developed since Wigner's pioneering work, few papers have been devoted to the problem of reducing the symmetrized nth powers of a corepresentation. The exceptions are Cracknell (1971) and Cracknell and Sedaghat (1972), containing the application of the cases $n = 2, 3$ to the theory of second-order phase transitions in magnetic crystals. The case $n = 2$ also has relevance to selection rules for magnetic materials (Backhouse 1974).

It is well known that a magnetic group $\boldsymbol{M}$ has a halving subgroup $\boldsymbol{G}$ of unitary elements and that many properties of $\boldsymbol{M}$ are deducible from the corresponding properties of these unitary elements. In what follows we assume a full knowledge of the representations of $\boldsymbol{G}$ in terms of which we show how to calculate and reduce the symmetrized nth powers of the corepresentations of $\boldsymbol{M}$. The first part of this problem, the determination of Clebsch–Gordan series, is discussed in § 2, relying on the result that a corepresentation is determined up to unitary equivalence by its restriction to the unitary subgroup. The second part of the programme, the determination of nth order symmetrized Clebsch–Gordan matrices, is resolved in § 3, and depends essentially on a construction of a complete reduction matrix for a reducible corepresentation from that of its associated ordinary representation. The generality of the latter offers an alternative procedure to that of Aviran and Litvin (1973) and Sakata (1974b) for the construction of Clebsch–Gordan matrices for corepresentations.

In connection with our work we draw attention to some recent papers devoted to or mentioning the reduction of symmetrized powers and the construction of Clebsch–Gordan matrices: Aviran and Zak (1968), Backhouse (1973), Backhouse and Gard (1974a, b, c), Boyle (1972), Bradley and Davies (1970), Cracknell (1974), Gard (1973a, b, c), Gard and Backhouse (1974), Koster (1958), Lewis (1973), Sakata (1974a).

2. Symmetrized powers of corepresentations

Employing the notation of Bradley and Davies (1968), we write the group of unitary and antiunitary elements as $\boldsymbol{M} = \boldsymbol{G} \cup A\boldsymbol{G}$, where $\boldsymbol{G}$ is the halving subgroup of unitary elements and A is an arbitrary antiunitary element. Let Δ be a UIR (unitary irreducible representation) of $\boldsymbol{G}$ and let D be the associated corep (irreducible corepresentation) of $\boldsymbol{M}$. The canonical form for the three types of coreps is derived, for example, by Bradley and Davies (1968). They show that if $\bar{\Delta}$, defined by $\bar{\Delta}(R) = \Delta^*(A^{-1}RA)$ for $R \in \boldsymbol{G}$, is unitarily equivalent to Δ, so that $\Delta(R) = P\bar{\Delta}(R)P^{-1}$ for all $R \in \boldsymbol{G}$, then the corresponding corep D may be taken in one of the following two forms.

Case (*a*)

If $PP^* = \Delta(A^2)$, then

$$D(R) = \Delta(R), \qquad \text{for } R \in \boldsymbol{G}$$

$$D(B) = \Delta(BA^{-1})P, \qquad \text{for } B \in A\boldsymbol{G}.$$

Case (*b*)

If $PP^* = -\Delta(A^2)$, then

$$D(R) = \begin{pmatrix} \Delta(R) & 0 \\ 0 & \Delta(R) \end{pmatrix} = \Delta(R) \dot{+} \Delta(R)$$

for $R \in \boldsymbol{G}$, where $\dot{+}$ denotes direct summation and is not to be confused with $+$ which denotes matrix addition;

$$D(B) = \begin{pmatrix} 0 & -\Delta(BA^{-1})P \\ \Delta(BA^{-1})P & 0 \end{pmatrix}, \qquad \text{for } B \in A\boldsymbol{G}.$$

If $\bar{\Delta}$ is not equivalent to Δ then case (c) gives the canonical form for D.

Case (*c*)

$$D(R) = \begin{pmatrix} \Delta(R) & 0 \\ 0 & \bar{\Delta}(R) \end{pmatrix} = \Delta(R) \dot{+} \bar{\Delta}(R) \qquad \text{for } R \in \boldsymbol{G};$$

$$D(B) = \begin{pmatrix} 0 & \Delta(BA) \\ \bar{\Delta}(BA^{-1}) & 0 \end{pmatrix} \qquad \text{for } B \in A\boldsymbol{G}.$$

Implicit in the above is that the equivalence class of a corep D is completely determined by the restriction of D to the unitary subgroup $\boldsymbol{G}$, which for the canonical forms is Δ, $\Delta \dotplus \Delta$ or $\Delta \dotplus \bar{\Delta}$. It follows that if $[v]$ denotes a UIR of $\boldsymbol{S}_n$, then the nth symmetrized power $D^{[v]}$ is determined by $\Delta^{[v]}$, $(\Delta \dotplus \Delta)^{[v]}$, $(\Delta \dotplus \bar{\Delta})^{[v]}$ in cases (a), (b), (c) respectively. We have here used the simple result that $D^{[v]} \downarrow \boldsymbol{G} = (D \downarrow \boldsymbol{G})^{[v]}$, where $\downarrow$ denotes restriction. It is worthwhile pointing out that there is at present no character theory for full corepresentations, so the above comments are essential. This point is discussed more fully by Cracknell (1971).

Since we assume we can symmetrize the UIR of $\boldsymbol{G}$, case (a) coreps can be dealt with immediately. The other cases require application of the following result, being a special case of the plethysm formula given by Littlewood (1950).

2.1. Lemma

Let Δ_1, Δ_2 be UIR of $\boldsymbol{G}$. Then

$$(\Delta_1 \dotplus \Delta_2)^{[v]} \equiv \sum \sigma(v; v_1, v_2) \Delta_1^{[v_1]} \otimes \Delta_2^{[v_2]}, \tag{2.1}$$

where the direct sum is taken over all partitions of n as $n = n_1 + n_2$ and over all UIR $[v_1]$, $[v_2]$ of $\boldsymbol{S}_{n_1}, \boldsymbol{S}_{n_2}$ respectively. $\sigma(v; v_1, v_2)$ is the frequency of $[v_1] \otimes [v_2]$ in $[v] \downarrow \boldsymbol{S}_{n_1} \times \boldsymbol{S}_{n_2}$.

In the lemma the number $\sigma(v; v_1, v_2)$, which is also the frequency of $[v]$ in $[v_1] \odot [v_2] = [v_1] \otimes [v_2] \uparrow \boldsymbol{S}_n$, is most easily calculated by using the diagram technique described by Hamermesh (1964). In particular we note the following results.

2.2. Corollary

1. $$(\Delta_1 \dotplus \Delta_2)^n \equiv \sum_{r=0}^{n} {}^nC_r (\Delta_1^r \otimes \Delta_2^{n-r}), \tag{2.2}$$

where ${}^nC_r$ is the binomial coefficient, $n!/(n-r)!r!$

2. $$(\Delta_1 \dotplus \Delta_2)^{[n]} \equiv \sum_{r=0}^{n} \Delta_1^{[r]} \otimes \Delta_2^{[n-r]}, \tag{2.3}$$

where $[k]$ is the totally symmetric rep of $\boldsymbol{S}_k$ and $\Delta^{[0]} = 1$.

3. $$(\Delta_1 \dotplus \Delta_2)^{[1^n]} \equiv \sum_{r=0}^{n} \Delta_1^{[1^r]} \otimes \Delta_2^{[1^{n-r}]}, \tag{2.4}$$

where $[1^k]$ is the totally antisymmetric rep of $\boldsymbol{S}_k$.

4. Let Δ_1, Δ_2 be linear characters and $[v] = [v_1, v_2]$, then

$$(\Delta_1 \dotplus \Delta_2)^{[v]} \equiv \sum_{r=v_2}^{v_1} \Delta_1^r \otimes \Delta_2^{n-r}. \tag{2.5}$$

From (2.3) and (2.4) we obtain for a case (c) corep the following equivalences:

$$(D \downarrow \boldsymbol{G})^{[2]} \equiv \Delta^{[2]} \dotplus (\Delta \otimes \bar{\Delta}) \dotplus \bar{\Delta}^{[2]}, \tag{2.6}$$

$$(D \downarrow \boldsymbol{G})^{[1^2]} \equiv \Delta^{[1^2]} \dotplus (\Delta \otimes \bar{\Delta}) \dotplus \bar{\Delta}^{[1^2]}, \tag{2.7}$$

$$(D \downarrow \boldsymbol{G})^{[3]} \equiv \Delta^{[3]} \dotplus (\Delta^{[2]} \otimes \bar{\Delta}) \dotplus (\Delta \otimes \bar{\Delta}^{[2]}) \dotplus \bar{\Delta}^{[3]}, \tag{2.8}$$

$$(D \downarrow \boldsymbol{G})^{[1^3]} \equiv \Delta^{[1^3]} \dotplus (\Delta^{[1^2]} \otimes \bar{\Delta}) \dotplus (\Delta \otimes \bar{\Delta}^{[1^2]}) \dotplus \bar{\Delta}^{[1^3]}. \tag{2.9}$$

These are results also given by Cracknell (1971), Cracknell and Sedaghat (1972). Also, either using (2.1) or using (2.2) with (2.8) and (2.9), we find

$$(D \downarrow G)^{[2,1]} \equiv \Delta^{[2,1]} \dotplus (\Delta^2 \otimes \bar{\Delta}) \dotplus (\Delta \otimes \bar{\Delta}^2) \dotplus \bar{\Delta}^{[2,1]}. \tag{2.10}$$

The corresponding results for case (b) are obtained by replacing $\bar{\Delta}$ with Δ. In using (2.1) it is often important to realise that $\Delta^{[\nu]}$ is void if dim Δ is less than the number of nonzero rows of $[\nu]$. Also, in applying (2.1) to coreps, it is useful to note its symmetry. Thus for case (b) coreps, for each term $\Delta^{[\nu_1]} \otimes \Delta^{[\nu_2]}$, there is also a term $\Delta^{[\nu_2]} \otimes \Delta^{[\nu_1]}$. For case ($c$) coreps, for each term $\Delta^{[\nu_1]} \otimes \bar{\Delta}^{[\nu_2]}$ there is also the conjugate term $\bar{\Delta}^{[\nu_1]} \otimes \Delta^{[\nu_2]}$. These observations approximately halve the amount of work involved in the reduction of $D^{[\nu]}$ for case (b), (c) coreps, and indeed they enable one to reproduce immediately table V of Clebsch–Gordan coefficients in Bradley and Davies (1968). It is important to note where and why our notation differs from that of Cracknell (1971) and Cracknell and Sedaghat (1972). These authors use square and brace brackets to denote totally symmetrized and totally antisymmetrized powers. Unfortunately the use of brackets does not easily extend to other symmetrized powers, hence our use of the superscript notation $D^{[\nu]}$ which is quite explicit.

3. The reduction of corepresentations

We have shown how the implementation of (2.1) allows a partial reduction of the symmetrized powers of case (b) and (c) coreps. Our assumed knowledge of the UIR of $\boldsymbol{G}$ implies that the corresponding Clebsch–Gordan series can be written down. However, the determination of complete reduction matrices requires more consideration. In any given example we may assume that we can explicitly write down a basis for the symmetrized corepresentation and hence explicitly determine the matrices which form the latter. For the determination of Clebsch–Gordan matrices it suffices to apply the solution of the following problem : given a corepresentation D of $\boldsymbol{M}$, find a unitary matrix W which satisfies

$$WD(m)\phi(m)W^{-1} = \sum_i a_i D_i(m) \tag{3.1}$$

for all $m \in \boldsymbol{M}$, where $\phi(m)$ is the identity operator if $m \in \boldsymbol{G}$ and is the complex conjugation operator if $m \in \boldsymbol{M}-\boldsymbol{G}$. The coreps D_i in (3.1) are assumed to be inequivalent fixed representatives of the equivalence classes of coreps, taken in canonical form.

The irreducible constituents of D, with their multiplicities a_i, are determined by those of $D \downarrow \boldsymbol{G}$, so let us assume that a unitary matrix U has been found where

$$U(D \downarrow \boldsymbol{G})U^{-1} = \sum_i a_i(D_i \downarrow \boldsymbol{G}), \tag{3.2}$$

in which $D_i \downarrow \boldsymbol{G}$ is Δ_i, $\Delta_i \dotplus \Delta_i$ or $\Delta_i \dotplus \bar{\Delta}_i$ according to type. U may be determined by the method of Koster (1958), Gard (1973c) or Sakata (1974a). Since U will not in general completely reduce D as well as $D \downarrow \boldsymbol{G}$, we must find a unitary matrix B such that $W = BU$. In (3.2) we see that B must preserve the reduced form of $D \downarrow \boldsymbol{G}$. Hence, dividing B into block matrices suitable for block matrix multiplication, it follows from Schur's lemma that each block is either zero or a scalar multiple of the identity. Thus $B = \Sigma_i B_i$, where $B_i = V_i \otimes I_i$, I_i is a unit matrix having the same dimension as Δ_i, and V_i is a unitary matrix whose dimensions depend on the canonical form of D_i. From this we deduce that any matrix U which reduces $D \downarrow G$ must reduce D itself to superblock form $\Sigma_i D^{(i)}$, where $D^{(i)} \downarrow G = a_i(D_i \downarrow G)$. Hence we seek matrices B_i such that

$$B_i D^{(i)} \phi B_i^{-1} = a_i D_i,$$

where $B_i = V_i \otimes I_i$. We consider the three types of coreps in turn, and for convenience we replace the index i by a subscript j ($j = 1, 2, 3$) to indicate the type of corep being considered.

Case (*a*)

Consider that summand of (3.1) corresponding to the case (*a*) corep D_1. Explicitly we have

$$a_1 D_1(R) = I_{a_1} \otimes \Delta_1(R) \tag{3.3}$$

$$a_1 D_1(A) = I_{a_1} \otimes P_1 \tag{3.4}$$

for $R \in G$, where I_{a_1} is the $a_1 \times a_1$ identity matrix and P_1 satisfies $P_1 P_1^* = \Delta_1(A^2)$. Let U be any unitary matrix which reduces $D \downarrow G$ to diagonal form. If $B = WU^{-1}$, where W reduces D to diagonal form, we know that the reduction of D by U must lead to the submatrix $(V_1^{*t} V_1^*) \otimes P_1$ instead of (3.4), where $B_1 = V_1 \otimes I_{d_1}$, $d_1 = \dim \Delta_1$ and V_1 is an $a_1 \times a_1$ unitary matrix. In an actual example we would reduce D, using U, to obtain the superblock $Q_1 \otimes P_1$ and then seek a unitary matrix V_1 such that $Q_1 = V_1^{*t} V_1^*$. This is done as follows.

Writing $T_1 = V_1^t$, then $Q_1 T_1 = T_1^*$ is the equation for the unknown unitary matrix T_1. Since Q_1 is necessarily unitary, there exist a_1 unitarily orthogonal eigenvectors $\{\boldsymbol{t}_k\}$ such that $Q_1 \boldsymbol{t}_k = \lambda_k \boldsymbol{t}_k$ where $|\lambda_k| = 1$. Since Q_1 is also symmetric, $Q_1^{-1} = Q_1^*$ and $Q_1 \boldsymbol{t}_k^* = \lambda_k \boldsymbol{t}_k^*$, so that $\boldsymbol{t}_k$ and $\boldsymbol{t}_k^*$ correspond to the same eigenvalue λ_k. If $\boldsymbol{t}_k$ and $\boldsymbol{t}_k^*$ are dependent then $\boldsymbol{t}_k = z_k \boldsymbol{e}_k$ where z_k is a complex number and $\boldsymbol{e}_k$ is a real eigenvector. Otherwise $\boldsymbol{t}_k + \boldsymbol{t}_k^*$ and $\mathrm{i}(\boldsymbol{t}_k - \boldsymbol{t}_k^*)$ are independent real eigenvectors. Hence we can construct a basis $\{\boldsymbol{e}_k\}$ of real orthogonal eigenvectors. Define $\boldsymbol{f}_k = \lambda_k^{-1/2} \boldsymbol{e}_k$, then $Q_1 \boldsymbol{f}_k = \lambda_k^{1/2} \boldsymbol{e}_k = \boldsymbol{f}_k^*$. We take T_i to be the unitary matrix built from the column vectors $\{\boldsymbol{f}_k\}$, then $B_1 = T_1^t \otimes I_{d_1}$.

Case (*b*)

The summand of (3.1) corresponding to the case (*b*) corep D_2 is given by

$$a_2 D_2(R) = (I_{a_2} \dotplus I_{a_2}) \otimes \Delta_2(R) \tag{3.5}$$

$$a_2 D_2(A) = I_{a_2} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes P_2, \tag{3.6}$$

where P_2 satisfies $P_2 P_2^* = -\Delta_2(A^2)$. Reducing D by $B^{-1}W$ leads to (3.5), but

$$V_2^{*t} \left[I_{a_2} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] V_2^* \otimes P_2$$

instead of (3.6), where $B_2 = V_2 \otimes I_{d_2}$, $d_2 = \dim \Delta_2$ and V_2 is a $2a_2 \times 2a_2$ unitary matrix.

It follows that if the reduction of $D(A)$ by U yields the component $Q_2 \otimes P_2$, we seek a unitary V_2 such that

$$Q_2 = V_2^{*\prime}\left[I_{a_2} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] V_2^*.$$

Write $V_2' = T_2$, then

$$Q_2 T_2 = T_2^* \left[I_{a_2} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right].$$

Since Q_2 is unitary there exists $2a_2$ unitarily orthogonal eigenvectors $\{s_k\}$ such that $Q_2 s_k = \rho_k s_k$, where $|\rho_k| = 1$. Also since Q_2 is skew-symmetric, $Q_2^{-1} = -Q_2^*$ and $Q_2 s_k^* = -\rho_k s_k^*$. It follows that we may choose a new basis such that the eigenvectors occur in complex conjugate pairs $\{u_k, u_k^*\}$. Now define $e_k^{(1)} = (\rho_k^{-1/2}/\sqrt{2})(u_k - u_k^*)$ and $e_k^{(2)} = (\rho_k^{-1/2}/\sqrt{2})(u_k + u_k^*)$. Then $Q_2 e_k^{(1)} = e_k^{(2)*}$ and $Q_2 e_k^{(2)} = -e_k^{(1)}$. Clearly we may take T_2 to be the unitary matrix built from pairs of column vectors $\{e_k^{(1)}, e_k^{(2)}\}$.

Case (c)

The canonical form of a case (c) corep D_3 is given by $D_3(R) = \Delta_3(R) \dotplus \bar{\Delta}_3(R)$ for $R \in G$, and

$$D_3(A) = \begin{pmatrix} 0 & \Delta_3(A^2) \\ I_{d_3} & 0 \end{pmatrix}, \tag{3.7}$$

where $d_3 = \dim \Delta_3$. However, it is convenient in cases of multiplicity to perform a simple permutation of the basis vectors so that the reduced part of D corresponding to the corep D_3 becomes $(I_{a_3} \otimes \Delta_3) \dotplus (I_{a_3} \otimes \bar{\Delta}_3)$ for the unitary part and

$$\begin{pmatrix} 0 & I_{a_3} \otimes \Delta_3(A^2) \\ I_{a_3} \otimes I_{d_3} & 0 \end{pmatrix} \tag{3.8}$$

for the representative of A. Reducing D by $B^{-1}W$ leads to the correct unitary part but gives

$$\begin{pmatrix} 0 & V_3^{*\prime}(1) V_3^*(2) \otimes \Delta_3(A^2) \\ V_3^{*\prime}(2) V_3^*(1) \otimes I_{d_3} & 0 \end{pmatrix} \tag{3.9}$$

instead of (3.8), where $B_3 = (V_3(1) \dotplus V_3(2)) \otimes I_{d_3}$ and $V_3(1)$, $V_3(2)$ are $a_3 \times a_3$ unitary matrices. Hence if the reduction of $D(A)$ by U yields the component

$$\begin{pmatrix} 0 & Q_3 \otimes \Delta_3(A^2) \\ Q_3' \otimes I_{d_3} & 0 \end{pmatrix}, \tag{3.10}$$

we seek unitary matrices $V_3(1)$, $V_3(2)$ such that $Q_3 = V_3^{*\prime}(1) V_3^*(2)$. The simplest solution is $V_3(1) = Q_3^{*\prime}$ and $V_3(2) = I_{a_3}$, and then $B_3 = (Q_3^{*\prime} \dotplus I_{a_3}) \otimes I_{d_3}$. Now applying the inverse of the original permutation, so that the final reduced form is given by (3.1), we obtain $B_3' = V_3 \otimes I_{d_3}$, where the submatrix $(V_3)_{ij}$ for i, j both odd is $(Q_3^{*\prime})_{ij}$, for i, j both even is $(I_{a_3})_{ij}$ and all other entries are zero.

To summarize: first find the unitary matrix U which completely reduces $D \downarrow G$ as in (3.2). Secondly, calculate the matrix $UD(A)U'$, which will automatically be partially reduced into superblocks, each superblock corresponding to one of the forms dealt with above. For each superblock we apply the relevant method to find the matrix B_i

(we have reverted to the original index i). The Clebsch–Gordan matrix is $W = (\Sigma_i B_i)U$ where the direct sum is over all superblocks.

We conclude with two very simple examples. The first illustrates the power of our method for finding ordinary Clebsch–Gordan coefficients, since the same example has been treated by Aviran and Zak (1968). Consider $\boldsymbol{M} = \boldsymbol{C}_{3V} + \theta \boldsymbol{C}_{3V}$ and the reduction of $D_3^* \otimes D_5$ to $D_4^* \dotplus D_4^*$, where in their notation D_3 is the two-dimensional single-valued corep and D_4, D_5 are the two-dimensional double-valued coreps of $\boldsymbol{M}$. The matrices of these coreps are given in table II of Aviran and Zak (1968), but for convenience we choose the angles α, ψ to be zero. It is easy to verify that the matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

reduces the unitary part of $D_3^* \otimes D_5$ to the unitary part of $D_4^* \dotplus D_4^*$, but leaves invariant the matrix representing θ, namely

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.12}$$

which can be rewritten as

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since D_4^* is case (a) we can write the complete reduction matrix $W = BU$, where $B = V \otimes I_2$ and V is a 2×2 unitary matrix satisfying

$$V^{*t} V^* = Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The theory gives

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}$$

and it is easy to check finally that W does indeed perform the complete reduction.

The second example illustrates the technique for finding the Clebsch–Gordan coefficients for a symmetrized power. For this we choose $\boldsymbol{M} = \boldsymbol{4}'$, having elements $\{E, C_2, \theta C_4, \theta C_4^3\}$, so that $\boldsymbol{G}$ is the subgroup $\{E, C_2\}$. It is easy to verify that the non-trivial rep of $\boldsymbol{G}$ extends to a case (b) corep of $\boldsymbol{M}$, and that the matrices of this corep are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for the elements E, C_2, θC_4, θC_4^3 respectively. Now the totally symmetrized cube of this corep, $D\Gamma_2$, has been computed by Cracknell and Sedaghat (1972) to be $D\Gamma_2 \dotplus D\Gamma_2$. Indeed we instantly verify this fact using equation (2.8). Again using (2.8) we see that if

$D\Gamma_2$ acts on the two-dimensional vector space with basis $\boldsymbol{u}, \boldsymbol{v}$, then $(D\Gamma_2)^{[3]}$ acts on the four-dimensional vector space with basis

$$\boldsymbol{u}\otimes\boldsymbol{u}\otimes\boldsymbol{u},\ \boldsymbol{u}\otimes\boldsymbol{u}\otimes\boldsymbol{v},\ \boldsymbol{u}\otimes\boldsymbol{v}\otimes\boldsymbol{v},\ \boldsymbol{v}\otimes\boldsymbol{v}\otimes\boldsymbol{v}.$$

On this space $(D\Gamma_2)^{[3]}(\theta C_4)$ is represented by the matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.13}$$

which is clearly not completely reduced. We are in a case (*b*) situation and we have $P = 1$ and $Q = X$, hence the theory tells us that we must seek a 4×4 unitary matrix V, where

$$Q = V^{*t}\left[I_2\otimes\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]V^{*},$$

and that then $B = V$ completely reduces $(D\Gamma_2)^{[3]}$. We find that Q has eigenvalues i, $-$i, i, $-$i, with corresponding eigenvectors $(1/\sqrt{2})(1\ 0\ 0\ -\mathrm{i})^t$, $(1/\sqrt{2})(1\ 0\ 0\ \mathrm{i})^t$, $(1/\sqrt{2})(0\ 1\ \mathrm{i}\ 0)^t$, $(1/\sqrt{2})(0\ 1\ -\mathrm{i}\ 0)^t$. Analysis quickly leads to

$$B = \begin{pmatrix} 0 & 0 & 0 & -\sqrt{\mathrm{i}} \\ 1/\sqrt{\mathrm{i}} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\mathrm{i}} & 0 \\ 0 & 1/\sqrt{\mathrm{i}} & 0 & 0 \end{pmatrix} \tag{3.14}$$

and we verify that B does indeed completely reduce $(D\Gamma_2)^{[3]}$.

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